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LIE-ALGEBRA OF THE ORTHOGONAL  
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## 1. INTRODUCTION

In recent years, the Lie-algebras of the orthogonal group, considered as associative algebras generated by a number of symbols satisfying a set of relations have been found to be of importance in the theory of elementary particles in Quantum Mechanics. The Clifford-Dirac algebra of 4 symbols which is the Lie-algebra of the orthogonal group in 5 dimensions was employed by Dirac in his theory of the electron whose spin is  $\frac{1}{2}$ . Explicit matrices of the representation of the Clifford-Dirac algebra with any number of symbols were given by Brauer and Weyl (1935). The Lie-algebra associated with an elementary particle of spin 1 was investigated by Kemmer (1939) and the matrices of the representations were obtained in an explicit form by D. E. Littlewood (1947).

The investigation of the Lie-algebras for higher spins is more complicated. It was, however, proved by Madhava Rao, Thiruvengatachar and Venkatachaliengar (1946) that the algebra for the case of half-odd-integral spins is the direct product of the corresponding Clifford-Dirac algebra and another algebra called the  $\xi$ -algebra generated by the symbols  $\xi_1, \xi_2, \dots, \xi_n$  satisfying the commutation rules

$$(i) \{ \xi_r, \{ \xi_s, \xi_t \} \} = \xi_s,$$

$$(ii) [ \xi_r, \{ \xi_s, \xi_t \} ] = 0; r \neq s \neq t,$$

where  $\{a, b\}$  is the anticommutator  $ab+ba$  and  $[a, b]$  is the commutator  $ab-ba$ .

This direct product resolution simplifies the problem of determining the irreducible representations of the Lie-algebra considerably. The matrices of the irreducible representations of the Lie-algebra are then the Kronecker products of the matrices of the irreducible representations of the  $\xi$ -algebra and those of the corresponding Clifford-Dirac algebra. In the case of spin  $\frac{3}{2}$ , the symbols generating the original Lie-algebra satisfy a quartic and the corresponding symbols  $\xi_r$  satisfy the quadratic

$$\xi^2 = \frac{3}{4} - \xi.$$

In this paper, we take up the investigation of the  $\xi$ -algebra  $A_n$  generated by the  $n$  symbols  $\xi_1, \xi_2, \dots, \xi_n$  with spin  $\frac{3}{2}$ . We show that the centre of the algebra is generated by a single element  $\theta$  and obtain the minimal equation it satisfies. We set  $\xi_{r-1} = \omega_{1r}$ ;  $\{\omega_{1r}, \omega_{1s}\} = \omega_{rs}$  and show that the irreducible representations of the algebra  $A_n$  are given by

I (a) when  $n$  is even ;

$$D_{nr}(\omega_{n,n+1}) = \frac{1}{2} E_{f_1} + \left| \begin{array}{cc} \frac{(2r-n-6)}{2(n-2r+4)} & 1 \\ \frac{(n-2r+4)^2-1}{(n-2r+4)^2} & \frac{(2r-n-2)}{2(n-2r+4)} \end{array} \right| \times E_{f_2} + \frac{1}{2} E_{f_3}$$

$$1 \leq r \leq \frac{n}{2} + 1$$

(b) when  $n$  is odd, we have the same expression for  $D_{nr}(\omega_{n,n+1})$  as I(a) with  $1 \leq r \leq \frac{n+1}{2}$  and an additional representation.

$$D_{n, \frac{n+3}{2}}(\omega_{n,n+1}) = \frac{1}{2} E_{f_4} + \frac{3}{2} E_{f_5}$$

where  $E_k$  is the unit matrix of order  $k$ .

$$\text{II. } D_{nr}(\omega_{p,p+1}) = D_{n-1,r-1}(\omega_{p,p+1}) + D_{n-1,r}(\omega_{p,p+1})$$

$$p = 1, 2, 3, \dots (n-1).$$

By taking the anticommutators of the  $\omega_{p,p+1}$  repeatedly, we obtain the matrices for  $\xi_r$ .

We prove also\* that the dimension of the algebra  $A_n$  is given by the simple expression  $\frac{2}{n+2} \binom{2n+1}{n}$ . It follows, therefore that the dimension of the corresponding Lie-algebra of the orthogonal group is given by

$$\frac{2^{n+1}}{n+2} \binom{2n+1}{n}.$$

## 2. THE $\xi$ -ALGEBRA

Let  $A_n$  be the  $\xi$ -algebra generated by the  $n$  symbols  $\xi_1, \xi_2, \dots, \xi_n$ , which satisfy the following relations:—

$$(1) (a). \quad \xi^2 = \frac{3}{4} - \xi^*$$

$$(1) (b). \quad \{\xi_r, \{\xi_r, \xi_s\}\} = \xi_s; \quad \{\xi_r, \xi_s\} = \xi_r \xi_s + \xi_s \xi_r.$$

$$(1) (c). \quad [\xi_r, \{\xi_s, \xi_t\}] = 0 \quad [\xi_r, \xi_s] = \xi_r \xi_s - \xi_s \xi_r$$

$$r \neq s \neq t.$$

We set

$$\xi_{r-1} = \omega_{1r} = \omega_{r1}; \quad [r = 2, 3, \dots (n+1)]$$

and

$$\{\xi_{r-1}, \xi_{s-1}\} = \{\omega_{1r}, \omega_{1s}\} = \omega_{rs}.$$

It follows easily that  $\{\xi_r, \xi_s\}^2 = \frac{3}{4} - \{\xi_r, \xi_s\}$ .

\* In the paper by Madhava Rao and collaborators referred to,  $\xi$  is taken to satisfy  $\xi^2 = \xi + \frac{3}{4}$ . We have here replaced  $\xi$  by  $-\xi$  for the slight simplification of some of the formulae.

Hence the  $\omega$ 's satisfy the relations :

$$\begin{aligned} (1') (a). \quad & \omega_{pq}^2 = \frac{3}{4} - \omega_{pq} \quad (p \neq q). \\ (1') (b). \quad & \{\omega_{1r}, \omega_{rs}\} = \omega_{1s}. \\ (1') (c). \quad & [\omega_{1r}, \omega_{st}] = 0; \quad \omega_{pq} = \omega_{qp}. \\ & r \neq s \neq t. \end{aligned}$$

From 1' (a), it follows that in any representation  $\omega_{pq}$  has the eigen-values  $\frac{1}{2}$  or  $-\frac{3}{2}$ . Consider a representation in which  $\omega_{1r}$  is diagonal, say  $|\lambda_r|$  and let  $\omega_{1s} = |a_{pq}|$ .

We have 
$$\{\omega_{1r}, \omega_{1s}\} = |(\lambda_p + \lambda_q) a_{pq}|.$$

Since 
$$\{\omega_{1r}, \omega_{rs}\} = \omega_{1s}$$

$$(\lambda_p + \lambda_q)^2 a_{pq} = a_{pq}$$

$$\therefore 4\lambda_p^2 a_{pp} = a_{pp},$$

i.e. either 
$$\lambda_p = \frac{1}{2} \text{ or } a_{pp} = 0.$$

Now 
$$\text{spur } \omega_{1s} = \sum a_{pp}$$

and 
$$\text{spur } \{\omega_{1r}, \omega_{1s}\} = \sum 2\lambda_p a_{pp} = \sum a_{pp} = \text{spur } \omega_{1s}.$$

Similarly, taking  $\omega_{1s}$  diagonal, we obtain

$$\text{spur } \omega_{1r} = \text{spur } \{\omega_{1s}, \omega_{1r}\}.$$

We therefore have

$$(2) \quad \text{spur } \omega_{1r} = \text{spur } \omega_{1s} = \text{spur } \omega_{rs}.$$

It is clear that (2) is true for any spin.

We now proceed to obtain explicit matrices for the irreducible representations of the algebra  $A_n$ . From the theory of the orthogonal group (cf. F. D. Murnaghan: The theory of group representations) it follows that the dimension formula for the irreducible representations of the Lie-algebra (generated by  $n$  symbols) is

$$(3) (a) \quad D_{\lambda_1 \lambda_2 \dots \lambda_k} = \frac{2^k l'_1 l'_2 \dots l'_k}{1 \cdot 3 \dots (2k-1)} \prod_{p < q}^k (l_p'^2 - l_q'^2); \quad n = 2k$$

and

$$(3) (b) \quad D_{\lambda_1 \lambda_2 \dots \lambda_k} = \frac{2^{k-1}}{2 \cdot 4 \dots (2k-2)} \prod_{p < q}^k (l_p^2 - l_q^2); \quad n = 2k-1,$$

where  $\lambda_1, \lambda_2, \dots, \lambda_k$  are half-odd integers such that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ ;  $l_p = \lambda_p + (k-p)$ ;  $l'_p = l_p + \frac{1}{2}$ ; and  $\lambda_1$  is the spin. Denoting by  $f_{\lambda_1 \lambda_2 \dots \lambda_k}$ , the dimension of the corresponding irreducible representation of the  $\xi$ -algebra, we have, since  $2^k$  (or  $2^{k-1}$ ) is the dimension of the irreducible representation of the Clifford-Dirac algebra according as  $n = 2k$  (or  $2k-1$ ),

$$(3') (a) \quad f_{\lambda_1 \lambda_2 \dots \lambda_k} = \frac{D_{\lambda_1 \lambda_2 \dots \lambda_k}}{2^k}; \quad n = 2k,$$

$$(3') (b) \quad f_{\lambda_1 \lambda_2 \dots \lambda_k} = \frac{D_{\lambda_1 \lambda_2 \dots \lambda_k}}{2^{k-1}}; \quad n = 2k-1.$$

If  $\text{spin} = \frac{3}{2}$ , the  $\lambda$ 's are all either  $\frac{3}{2}$  or  $\frac{1}{2}$ .

Let  $f_{nr} \equiv f_{\frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}}$  with  $(r-1) \rightarrow \frac{3}{2}$ 's and  $(k-r+1) \rightarrow \frac{1}{2}$ 's,  $r$  taking the values  $1, 2, \dots, (k+1)$  where  $n = 2k$  or  $2k-1$ ; there will be  $(k+1)$  irreducible representations of the algebra  $A_n$ .

We obtain after some simplification that the formulae 3'(a) and 3'(b) both reduce to

$$(4) \quad f_{nr} \equiv f_{\frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}} = \frac{n-2r+4}{n+2} \binom{n+2}{r-1}$$

we define  $f_{nr} = 0$  for  $r \geq \frac{n+4}{2}$ , since  $f_{nr} = 0$  for  $n = 2r-4$  and negative for  $n < 2r-4$ . i.e.  $r$  takes the values

$$1, 2, 3, \dots, \frac{n}{2} + 1 \text{ if } n \text{ is even}$$

and

$$1, 2, 3, \dots, \frac{n+1}{2}, \frac{n+3}{2} \text{ if } n \text{ is odd.}$$

We denote by  $D_{nr}$ , the irreducible representation (of the algebra  $A_n$ ) whose dimension is  $f_{nr}$  and by  $D_{nr}(\omega_{pq})$  the matrix of representation for  $\omega_{pq}$  in the representation  $D_{nr}$ .

It follows from the theory of the orthogonal group that the algebra  $A_n$  branches over the algebra  $A_{n-1}$  according to the law

$$(5) \quad D_{nr}(\omega_{pq}) = D_{n-1, r-1}(\omega_{pq}) + D_{n-1, r}(\omega_{pq})$$

[It is verified easily that  $f_{nr} = f_{n-1, r-1} + f_{n-1, r}$ ].

We next show that if  $S_{nr}$  is the spur of any  $\omega_{pq}$  in the irreducible representation  $D_{nr}$ , then

$$(6) \quad \begin{aligned} S_{nr} &= f_{nr} \frac{n^2 + (5-4r)n + 4(r-1)(r-3)}{2n(n+1)} \\ &= (n-2r+4) \frac{(n-1)(n-2) \dots (n-r+4)}{2 \lfloor r-1 \rfloor} \times \\ &\quad \times \{n^2 + (5-4r)n + 4(r-1)(r-3)\} \end{aligned}$$

*Proof:*

We assume the result to be true for the algebra  $A_{n-1}$  and prove it for  $A_n$ . From (5) it follows that

$$S_{nr} = S_{n-1, r-1} + S_{n-1, r}.$$

Now  $S_{n-1, r-1} + S_{n-1, r} =$

$$\begin{aligned} &= \frac{(n-2r+5)}{2 \lfloor r-2 \rfloor} (n-2)(n-3) \dots (n-r+4) \{n^2 + (7-4r)n + 4(r-2)(r-3)\} \\ &\quad + \frac{(n-2r+3)}{2 \lfloor r-1 \rfloor} (n-2)(n-3) \dots (n-r+3) \{n^2 + (3-4r)n + 4(r-1)(r-2)\} \\ &= \frac{(n-2) \dots (n-r+4)}{2 \lfloor r-1 \rfloor} \left[ \frac{(n-2r+5)(r-1)}{(n-2r+3)(n-r+3)} \{n^2 + (7-4r)n + 4(r-2)(r-3)\} + \right. \\ &\quad \left. \{n^2 + (3-4r)n + 4(r-1)(r-2)\} \right] \end{aligned}$$

i.e.  $S_{n-1, r-1} + S_{n-1, r} =$

$$= \frac{(n-2) \dots (n-r+4)}{2 \lfloor r-1 \rfloor} \left[ (n-2r+4)(n-1) \{n^2 + (5-4r)n + 4(r-1)(r-3)\} \right]$$

on simplification.

$$= S_{nr}.$$

Now  $D_{n1}(\omega_{pq}) = \frac{1}{2}$  for all  $n$  so that  $S_{n1} = \frac{1}{2}$  and for the algebra  $A_2$ , one can see easily that

$$D_{22}^*(\omega_{12}) = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} \end{vmatrix} \text{ or } S_{22} = -1.$$

That is, the formula is true for  $S_{n1}$  and  $S_{22}$  and hence by induction it is universally true.

### 3. THE IRREDUCIBLE REPRESENTATIONS OF $A_n$

We consider the algebra  $A_n$  as generated by the  $n$  symmetric symbols  $\omega_{12}, \omega_{23}, \dots, \omega_{p, p+1}, \dots, \omega_{n, n+1}$  and define

$$\omega_{rs} = \{\omega_{1r}, \omega_{1s}\}; r \neq s \text{ with the relations (1')}.$$

From (1') (c) it follows that

$$[\omega_{pq}, \omega_{st}] = 0; \quad p, q \neq s, t.$$

Therefore,  $\omega_{n, n+1}$  commutes with  $\omega_{12}, \omega_{23}, \dots, \omega_{n-2, n-1}$ , i.e. with the algebra  $A_{n-2}$ . We also have the branching law

$$D_{nr}(\omega_{p, p+1}) = D_{n-1, r-1}(\omega_{p, p+1}) + D_{n-1, r}(\omega_{p, p+1}); \quad p = 1, 2, 3, \dots, (n-1).$$

We now show that—

(7) (a). When  $n$  is even

$$D_{nr}(\omega_{n, n+1}) = \frac{1}{2} E_{f_1} + \begin{vmatrix} \frac{2r-n-6}{2(n-2r+4)} & 1 \\ \frac{(n-2r+4)^2-1}{(n-2r+4)^2} & \frac{(2r-n-2)}{2(n-2r+4)} \end{vmatrix} \times E_{f_2} + \frac{1}{2} E_{f_3}$$

$$1 \leq r \leq \frac{n}{2} + 1.$$

(7) (b). When  $n$  is odd, we have the same expression for  $D_{nr}(\omega_{n, n+1})$  as (7) (a) with

$1 \leq r \leq \frac{n+1}{2}$  and an additional representation

$$(8) \quad D_{n, \frac{n+3}{2}}(\omega_{n, n+1}) = \frac{1}{2} E_{f_4} + \frac{3}{2} E_{f_5},$$

where  $E_f, E_{f_2}, E_{f_3}, E_{f_4}, E_{f_5}$  are unit matrices of orders

$$f_1 = f_{n-2, r-2}$$

$$f_2 = f_{n-2, r-1}$$

$$f_3 = f_{n-2, r}$$

$$f_4 = f_{n-2, \frac{n-1}{2}}$$

$$f_5 = f_{n-2, \frac{n+1}{2}} \text{ respectively}$$

with  $E_k = 0$  for  $k \leq 0$  and  $E_1 = 1$ .

*Proof:*

We first of all determine  $D_{n, \frac{n+3}{2}}(\omega_{n, n+1})$  when  $n$  is odd. We have, by the branching law, that  $D_{n, \frac{n+3}{2}}$  is the same as  $D_{n-1, \frac{n+1}{2}}$  over the algebra  $A_{n-1}$ . Taking the branching again over  $A_{n-2}$ , we have

$$D_{n, \frac{n+3}{2}}(\omega_{p, p+1}) = D_{n-2, \frac{n-1}{2}}(\omega_{p, p+1}) + D_{n-2, \frac{n+1}{2}}(\omega_{p, p+1}) \text{ over } A_{n-2};$$

$$(1 \leq p \leq n-2).$$

Since  $\omega_{n, n+1}$  commutes with the algebra  $A_{n-2}$ , we have by the Schur lemma,

$$D_{n, \frac{n+3}{2}}(\omega_{n, n+1}) = \lambda_1 E_{f_4} + \lambda_2 E_{f_5}.$$

Writing  $n = 2m+1$ , we have

$$f_4 = f_{2m-1, m} = \frac{3(m+3)(m+4) \dots (2m)}{|m-1|}$$

$$f_5 = f_{2m-1, m+1} = \frac{(m+2)(m+3) \dots (2m)}{|m|}.$$

Hence taking the spur of  $\omega_{n, n+1}$ , we have

$$(9) \quad \lambda_1 f_4 + \lambda_2 f_5 = S_{n, \frac{n+3}{2}} = S_{2m+1, m+2}$$

$$= \frac{-3(m+3)(m+4) \dots (2m)}{|m|},$$

i.e.

$$\lambda_1 3m + \lambda_2(m+2) = -3$$

or

$$\lambda_1 = \frac{1}{2} \text{ and } \lambda_2 = -\frac{3}{2}.$$

Thus, we have

$$(8) \quad D_{n, \frac{n+3}{2}}(\omega_{n, n+1}) = \frac{1}{2} E_{f_4} - \frac{3}{2} E_{f_5}.$$

To prove (7), we observe, first of all by the branching law, that

$$D_{nr}(\omega_{p, p+1}) = D_{n-1, r-1}(\omega_{p, p+1}) + D_{n-1, r}(\omega_{p, p+1}) \text{ over } A_{n-1}$$

$$= D_{n-2, r-2}(\omega_{p, p+1}) + D_{n-2, r-1}(\omega_{p, p+1})$$

$$+ D_{n-2, r-1}(\omega_{p, p+1}) + D_{n-2, r}(\omega_{p, p+1}) \text{ over } A_{n-2}$$

$$= D_{n-2, r-2}(\omega_{p, p+1}) + E_2 \times D_{n-2, r-1}(\omega_{p, p+1})$$

$$+ D_{n-2, r}(\omega_{p, p+1}) \text{ over } A_{n-2}.$$

Since  $\omega_{n,n+1}$  commutes with the algebra  $A_{n-2}$ , we have by the Schur lemma,

$$D_{nr}(\omega_{n,n+1}) = a_{11} E_{f_1} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \times E_{f_2} + a_{44} E_{f_3},$$

where

$$f_1 = f_{n-2, r-2}; f_2 = f_{n-2, r-1}; f_3 = f_{n-2, r}.$$

Since

$$\omega_{n,n+1}^2 = \frac{3}{4} - \omega_{n,n+1},$$

$$a_{11}, a_{44} = \frac{1}{2} \text{ or } -\frac{3}{2}; a_{22} + a_{33} = -1$$

and

$$a_{22}^2 + a_{23} a_{32} = \frac{3}{4} - a_{22}.$$

Taking the spur of  $\omega_{n,n+1}$ , we have

$$a_{11} f_1 + (a_{22} + a_{33}) f_2 + a_{44} f_3 = S_{nr},$$

i.e.

$$\begin{aligned} (10) \quad & \frac{(n-r+4)(n-r+5) \dots (n-1)}{r-3} \left\{ a_{11} (n-2r+6) - \frac{(n-2r+4)(n-r+3)}{r-2} \right. \\ & \left. + a_{44} \frac{(n-2r+2)(n-r+2)(n-r+3)}{(r-1)(r-2)} \right\} \\ & = (n-2r+4) \frac{(n-r+4)(n-r+5) \dots (n-1)}{2(r-1)} \times \\ & \quad \times \{ n^2 + (5-4r)n + 4(r-1)(r-3) \}. \end{aligned}$$

It is easily seen that (10) will be consistent for the value  $\frac{1}{2}$  only for  $a_{11}$  and  $a_{44}$ .

We now assume the result (7) for  $n = m$  and prove it for  $(m+1)$ .

We have just proved that

$$D_{m+1, r}(\omega_{m+1, m+2}) = \frac{1}{2} E_{f'_1} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \times E_{f'_2} + \frac{1}{2} E_{f'_3},$$

where

$$f'_1 = f_{m-1, r-2}; f'_2 = f_{m-1, r-1}; f'_3 = f_{m-1, r} \text{ and } a_{22} + a_{33} = -1.$$

Hence

$$\text{spur}(\omega_{m+1, m+2}) = \frac{1}{2} f'_1 - f'_2 + \frac{1}{2} f'_3 = S_{m+1, r}.$$

Now

$D_{m+1, r}(\omega_{m, m+1}) = D_{m, r-1}(\omega_{m, m+1}) + D_{m, r}(\omega_{m, m+1})$  by the branching law

$$\begin{aligned} & = \frac{1}{2} E_{f''_1} + \begin{vmatrix} \frac{(2r-m-8)}{2(m-2r+6)} & 1 \\ \frac{(m-2r+6)^2-1}{(m-2r+1)^2} & \frac{(2r-m-4)}{2(m-2r-6)} \end{vmatrix} \times E_{f''_2} + \frac{1}{2} E_{f''_3} \\ & + \frac{1}{2} E_{f'_1} + \begin{vmatrix} \frac{(2r-m-6)}{2(m-2r+4)} & 1 \\ \frac{(m-2r+4)^2-1}{(m-2r+4)^2} & \frac{(2r-m-2)}{2(m-2r+4)} \end{vmatrix} \times E_{f'_2} + \frac{1}{2} E_{f'_3}, \end{aligned}$$

where

$$\begin{aligned} f_1'' &= f_{m-2, r-3}; \quad f_2'' = f_{m-2, r-2}; \quad f_3'' = f_{m-2, r-1} \\ f_1 &= f_{m-2, r-2}; \quad f_2 = f_{m-2, r-1}; \quad f_3 = f_{m-2, r}. \end{aligned}$$

It follows easily from (1')(b) and (1')(c) that  $\{\omega_{rs}, \omega_{st}\} = \omega_{rt}$  and hence  $\text{spur } \{\omega_{m, m+1}, \omega_{m+1, m+2}\}$  is also  $S_{m+1, r}$ ; we thus have

$$\begin{aligned} (11) \quad & \frac{1}{2} f_1'' - \frac{(m-2r+8)}{2(m-2r+6)} f_2'' - \frac{(m-2r+4)}{(m-2r+6)} a_{22} f_2'' \\ & + a_{22} f_3'' + a_{33} f_1 - \frac{(m-2r+6)}{(m-2r+4)} a_{33} f_2 \\ & - \frac{(m-2r+2)}{2(m-2r+4)} f_2 + \frac{1}{2} f_3 = S_{m+1, r} \end{aligned}$$

[Observe that  $f_1' - f_1'' = f_2''$ ;  $f_2' - f_2'' = f_3''$ ;  $f_2' - f_1 = f_2$ ;  $f_3' - f_2 = f_3$ ].

We also have

$$(12) \quad a_{22} + a_{33} = -1.$$

On solving for  $a_{22}$  and  $a_{33}$  from (11) and (12), we obtain,

$$a_{22} = \frac{2r-m-7}{2(m-2r+5)}$$

$$a_{33} = \frac{2r-m-3}{2(m-2r+5)}$$

from  $a_{22}^2 + a_{32} a_{23} = \frac{3}{4} - a_{22}$ , we have

$$a_{32} a_{23} = \frac{(m-2r+5)^2 - 1}{(m-2r+5)^2}.$$

If  $a_{22} = \frac{1}{2}$  or  $-\frac{3}{2}$ ,  $r = \frac{m+6}{2}$  or  $\frac{m+4}{2}$  respectively and this is clearly not possible.

Hence  $a_{22} \neq \frac{1}{2}$  or  $-\frac{3}{2}$  or  $a_{32} a_{23} \neq 0$ .

We now effect a similarity transformation of the matrices  $D_{m+1, r}(\omega_{m+1, m+2})$  and  $D_{m+1, r}(\omega_{m, m+1})$  by the matrix  $\frac{1}{a_{23}} E_{f_{m, r-1}} + E_{f_{m, r}}$ . This leaves  $D_{m+1, r}(\omega_{m, m+1})$  unaltered while in  $D_{m+1, r}(\omega_{m+1, m+2})$  it changes  $a_{23}$  to 1 and  $a_{32}$  to  $a_{32} a_{23}$ . We have thus proved the result for  $n = m+1$  if it is true for  $n = m$ . Before completing the proof by induction, we observe that the foregoing does not cover the case  $r = \frac{n}{2} + 1$  when  $n$  is even; for then

$$\begin{aligned} D_{n, \frac{n+2}{2}}(\omega_{n-1, n}) &= D_{n-1, \frac{n}{2}}(\omega_{n-1, n}) + D_{n-1, \frac{n+2}{2}}(\omega_{n-1, n}) \\ &= D_{n-1, \frac{n}{2}}(\omega_{n-1, n}) + \frac{1}{2} E_{f_{n-3, \frac{n-2}{2}}} + -\frac{3}{2} E_{f_{n-3, \frac{n}{2}}} \end{aligned}$$

We therefore treat this case separately: i.e. writing  $n = 2p$ , we show that

$$D_{2p, p+1}(\omega_{2p, 2p+1}) = \frac{1}{2} E_{f_{2p-2, p-1}} + \begin{vmatrix} -1 & 1 \\ \frac{3}{4} & 0 \end{vmatrix} \times E_{f_{2p-2, p}}.$$

We assume the result for  $n = 2p$  and prove it for  $2p+2$ . The preceding result shows that (7) is true for  $n = 2p+1$ .

Now

$$\begin{aligned} D_{2p+2, p+2}(\omega_k, k+1) &= D_{2p+1, p+1}(\omega_k, k+1) + D_{2p+1, p+2}(\omega_k, k+1) \text{ over } A_{n-1} \\ &= D_{2p, p}(\omega_k, k+1) + D_{2p, p+1}(\omega_k, k+1) + D_{2p, p+1}(\omega_k, k+1) \text{ over } A_{n-2}. \end{aligned}$$

Hence from the Schur lemma,

$$D_{2p+2, p+2}(\omega_{2p+2, 2p+3}) = a_{11} E_{f_{2p, p}} + \begin{vmatrix} a_{22} & 1 \\ a_{32} & a_{33} \end{vmatrix} \times E_{f_{2p, p+1}}$$

where  $a_{23}$  is taken to be 1 as before.

From  $\omega_{2p+2, 2p+3}^2 = \frac{3}{4} - \omega_{2p+2, 2p+3}$ , we have again  $a_{11} = \frac{1}{2}$  or  $-\frac{3}{2}$  and  $a_{22} + a_{33} = -1$ .

Taking the spur of  $\omega_{2p+2, 2p+3}$ , we have

$$\begin{aligned} a_{11} \frac{4(p+4)(p+5) \dots (2p+1)}{|p-1|} - \frac{2(p+3)(p+4) \dots (2p+1)}{|p|} &= S_{2p+2, p+2} \\ &= -6 \frac{(p+1)(p+4)(p+5) \dots (2p+1)}{|p+1|} \end{aligned}$$

from which  $a_{11} = \frac{1}{2}$  only. Therefore,

$$D_{2p+2, p+2}(\omega_{2p+2, 2p+3}) = \frac{1}{2} E_{f_{2p, p}} + \begin{vmatrix} a_{22} & 1 \\ a_{32} & a_{23} \end{vmatrix} \times E_{f_{2p, p+1}}$$

and

$$D_{2p+2, p+2}(\omega_{2p+1, 2p+2}) = D_{2p+1, p+1}(\omega_{2p+1, 2p+2}) + D_{2p+1, p+2}(\omega_{2p+1, 2p+2}),$$

i.e.

$$\begin{aligned} D_{2p+2, p+2}(\omega_{2p+1, 2p+2}) &= \\ &= \frac{1}{2} E_{f_{2p-1, p-1}} + \begin{vmatrix} -\frac{5}{6} & 1 \\ \frac{8}{9} & -\frac{1}{6} \end{vmatrix} \times E_{f_{2p-1, p}} + \frac{1}{2} E_{f_{2p-1, p+1}} \\ &\quad + \frac{1}{2} E_{f_{2p-1, p}} + -\frac{3}{2} E_{f_{2p-1, p+1}}. \end{aligned}$$

Since spur of  $\{\omega_{2p+1, 2p+2}, \omega_{2p+2, 2p+3}\}$  is also  $S_{2p+2, p+2}$  we have

$$\begin{aligned} \frac{1}{2} f_{2p-1, p-1} - \frac{5}{6} f_{2p-1, p} - \frac{1}{3} f_{2p-1, p} a_{22} + f_{2p-1, p+1} a_{22} + f_{2p-1, p} a_{33} - 3 f_{2p-1, p+1} \\ = S_{2p+2, p+2} = \frac{1}{2} f_{2p, p} - f_{2p, p+1} \end{aligned}$$

and we also have  $a_{22} + a_{33} = -1$ . On solving for  $a_{22}, a_{33}$ , we obtain

$$a_{22} = -1, a_{33} = 0; a_{32} = \frac{3}{4}.$$

We have thus shown that in all cases (7) is true for  $n = m+1$  if it is true for  $n = m$ . Now for the algebra  $A_2$ , one can show easily that

$$D_{22}(\omega_{12}) = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{3}{2} \end{vmatrix}; D_{22}(\omega_{23}) = \begin{vmatrix} -1 & 1 \\ \frac{3}{4} & 0 \end{vmatrix}$$

and  $D_{n1}(\omega_{p, p+1}) = \frac{1}{2}$  for all  $n$ . This proves that the irreducible representations of  $A_n$  are given by (7) and in case  $n$  is odd, we have an additional representation given by (8)

We observe that the representation matrices are chosen in such a way that their elements are rational numbers. If, however, we want them to be symmetric matrices as is generally required in Quantum Mechanics we can take

$$a_{23} = a_{32} = \sqrt{\frac{(n-2r+4)^2-1}{(n-2r+4)^2}}.$$

As an illustration, we give below the matrices of the irreducible representations for  $\omega_{12}, \omega_{23}, \dots, \omega_{56}$  of the algebra  $A_5$ . By taking the anti-commutators of these  $\omega_{p, p+1}$  repeatedly we can compute the matrices for  $\xi_{r-1} = \omega_{1r}$ .

The algebra  $A_5$  has 4 irreducible representations  $D_{51}, D_{52}, D_{53}, D_{54}$  of orders 1, 5, 9, 5 respectively.

(i)  $D_{51}$  :—

$$\omega_{r, r+1} = \frac{1}{2}; \quad r = 1, 2, 3, 4, 5.$$

(ii)  $D_{52}$  :—

$$\omega_{12} = \frac{1}{2} E_4 + \frac{3}{2} E_1$$

$$\omega_{23} = \frac{1}{2} E_3 + \begin{vmatrix} -1 & 1 \\ \frac{3}{4} & 0 \end{vmatrix}$$

$$\omega_{34} = \frac{1}{2} E_2 + \begin{vmatrix} -\frac{5}{6} & 1 \\ \frac{8}{9} & -\frac{1}{6} \end{vmatrix} + \frac{1}{2} E_1$$

$$\omega_{45} = \frac{1}{2} E_1 + \begin{vmatrix} -\frac{3}{4} & 1 \\ \frac{15}{16} & -\frac{1}{4} \end{vmatrix} + \frac{1}{2} E_2$$

$$\omega_{56} = \begin{vmatrix} -\frac{7}{10} & 1 \\ \frac{24}{25} & -\frac{3}{10} \end{vmatrix} + \frac{1}{2} E_3$$

(iii)  $D_{53}$  :—

$$\omega_{12} = \frac{1}{2} E_3 + \frac{3}{2} E_1 + \frac{1}{2} E_2 + \frac{3}{2} E_1 + \frac{1}{2} E_1 + \frac{3}{2} E_1$$

$$\omega_{23} = \frac{1}{2} E_2 + \begin{vmatrix} -1 & 1 \\ \frac{3}{4} & 0 \end{vmatrix} + \frac{1}{2} E_1 + \begin{vmatrix} -1 & 1 \\ \frac{3}{4} & 0 \end{vmatrix} \times E_2$$

$$\omega_{34} = \frac{1}{2} E_1 + \begin{vmatrix} -\frac{5}{6} & 1 \\ \frac{8}{9} & -\frac{1}{6} \end{vmatrix} + \frac{1}{2} E_1 + \begin{vmatrix} -\frac{5}{6} & 1 \\ \frac{8}{9} & -\frac{1}{6} \end{vmatrix} + \frac{1}{2} E_2 + \frac{3}{2} E_1$$

$$\omega_{45} = \begin{vmatrix} -\frac{3}{4} & 1 \\ \frac{15}{16} & -\frac{1}{4} \end{vmatrix} + \frac{1}{2} E_3 + \begin{vmatrix} -1 & 1 \\ \frac{3}{4} & 0 \end{vmatrix} \times E_2$$

$$\omega_{56} = \frac{1}{2} E_1 + \begin{vmatrix} -\frac{5}{6} & 1 \\ \frac{8}{9} & -\frac{1}{6} \end{vmatrix} \times E_3 + \frac{1}{2} E_2$$

(iv)  $D_{54}$  :—

$$\omega_{12} = \frac{1}{2} E_2 + -\frac{3}{2} E_1 + \frac{1}{2} E_1 + -\frac{3}{2} E_1$$

$$\omega_{23} = \frac{1}{2} E_1 + E_2 \times \begin{vmatrix} -1 & 1 \\ \frac{3}{4} & 0 \end{vmatrix}$$

$$\omega_{34} = \begin{vmatrix} -\frac{5}{6} & 1 \\ \frac{5}{6} & -\frac{1}{6} \end{vmatrix} + \frac{1}{2} E_2 + -\frac{3}{2} E_1$$

$$\omega_{45} = \frac{1}{2} E_1 + \begin{vmatrix} -1 & 1 \\ \frac{3}{4} & 0 \end{vmatrix} \times E_2$$

$$\omega_{56} = \frac{1}{2} E_3 + -\frac{3}{2} E_2.$$

#### 4. THE DIMENSION OF THE ALGEBRA $A_n$

We now prove that the dimension of the algebra  $A_n$  is given by the simple

expression  $\frac{2}{n+2} \binom{2n+1}{n}$ ; i.e. we show that

$$\sum_{r=1}^{m+1} f_{nr}^2 = \sum_{r=1}^{m+1} \left( \frac{n-2r+1}{n+2} \right)^2 \binom{n+2}{r-1}^2 = \frac{2}{n+2} \binom{2n+1}{n}$$

for  $n = 2m$  or  $2m-1$

(i) Let  $n = 2m$ ; to show that

$$\sum_0^m (m-p+1)^2 \binom{2m+2}{p}^2 = (m+1) \binom{4m+1}{2m}$$

Proof :—

$$(1+x)^{2m+2} = \sum_0^{2m+2} \binom{2m+2}{2m+2-p} x^{2m+2-p}.$$

$$\therefore x^{-m-1} (1+x)^{2m+2} = \sum_0^{2m+2} \binom{2m+2}{2m+2-p} x^{m-p+1}$$

$$\therefore \frac{d}{dx} \{ x^{-m-1} (1+x)^{2m+2} \} = \sum_0^{2m+2} (m-p+1) \binom{2m+2}{2m+2-p} x^{m-p}$$

Also

$$(1+x)^{2m+2} = \sum_0^{2m+2} \binom{2m+2}{p} x^p.$$

$$\therefore \frac{d}{dx} \{ x^{-m-1} (1+x)^{2m+2} \} = - \sum_0^{2m+2} (m-p+1) \binom{2m+2}{p} x^{p-m-2}$$

Hence, in

$$- \left\{ \sum_0^{2m+2} (m-p+1) \binom{2m+2}{2m+2-p} x^{m-p} \right\} \left\{ \sum_0^{2m+2} (m-p+1) \binom{2m+2}{p} x^{p-m-2} \right\}$$

the coefficient of  $\frac{1}{x^2} = - \sum_0^{2m+2} (m-p+1)^2 \binom{2m+2}{p}^2$

$$= -2 \sum_0^m (m-p+1)^2 \binom{2m+2}{p}^2.$$

This must therefore be equal to the coefficient of  $\frac{1}{x^2}$  in

$$\left[ \frac{d}{dx} \left\{ x^{-m-1} (1+x)^{2m+2} \right\} \right]^2,$$

i.e. to the coefficient of  $\frac{1}{x^2}$  in

$$\left\{ (2m+2) (1+x)^{2m+1} x^{-m-1} - (m+1) (1+x)^{2m+2} x^{-m-2} \right\}^2$$

$$= \frac{(m+1)^2 (1+x)^{4m+2} (1-x)^2}{x^{2m+4}}.$$

Coefficient of  $\frac{1}{x^2} = (m+1)^2 \left\{ \binom{4m+2}{2m+2} - 2 \binom{4m+2}{2m+1} + \binom{4m+2}{2m} \right\}$

$$= -2(m+1) \binom{4m+1}{2m} \text{ on simplification.}$$

Hence

$$\sum_0^m (m-p+1)^2 \binom{2m+2}{p}^2 = (m+1) \binom{4m+1}{2m}.$$

When  $n = 2m-1$ , one can similarly prove the result by considering the expansion for  $(1+x^2)^{2m+1}$ . We thus obtain that the dimension of the Lie-algebra of the orthogonal group with spin  $\frac{3}{2}$  is

$$\frac{2^{n+1}}{n+2} \binom{2n+1}{n}.$$

## 5. THE CENTRE OF THE ALGEBRA $A_n$

Let  $P_r = \sum \xi_{p_1 p_2 \dots p_r}$  where  $\xi_{p_1 p_2 \dots p_r} = \xi_{p_1} \xi_{p_2} \dots \xi_{p_r}$  and the summation extends over  $n P_r$  permutations.

Thus  $\xi_1 P_{2m+1} = \sum \xi_1 \xi_{q_1 q_2 \dots q_{2m+1}} + \sum \xi_1 \xi_{1 q_2 \dots q_{2m+1}} + \sum \xi_1 \xi_{q_2 1 q_3 \dots q_{2m+1}}$

$$+ \dots + \dots + \sum \xi_1 \xi_{q_2 \dots q_{2m+1} 1}$$

where the  $q_r$  are to be summed up over all indices excepting 1.

Since  $[\xi_1, \{\xi_{q_1}, \xi_{q_2}\}] = 0$ , we obtain

$$\xi_1 P_{2m+1} = \xi_{1q_1 \dots q_{2m+1}} + (m+1) \xi_{11q_2 \dots q_{2m+1}} + m \xi_{1q_2 1q_3 \dots q_{2m+1}},$$

the  $q_r$  being summed up over all indices excepting 1.

Now

$$2 \xi_{1q_1} + \xi_{q_1 1} = \xi_q - \xi_{11q}$$

$$\begin{aligned} \therefore \xi_1 P_{2m+1} &= \xi_{1q_1 q_2 \dots q_{2m+1}} + \frac{m}{2} \xi_{q_2 q_3 \dots q_{2m+1}} \\ &\quad - \frac{m}{2} \xi_{q_2 11q_3 \dots q_{2m+1}} + \left(\frac{m}{2} + 1\right) \xi_{11q_2 \dots q_{2m+1}}. \end{aligned}$$

From

$$\xi_{11} = \frac{3}{4} - \xi_1 \quad \text{we obtain}$$

$$\begin{aligned} \xi_1 P_{2m+1} &= \xi_{1q_1 \dots q_{2m+1}} + \frac{2m+3}{4} \xi_{q_2 \dots q_{2m+1}} \\ &\quad - \frac{m+2}{2} \xi_{1q_2 \dots q_{2m+1}} + \frac{m}{2} \xi_{q_2 1q_3 \dots q_{2m+1}} \end{aligned}$$

we have similarly

$$\begin{aligned} P_{2m+1} \xi_1 &= \xi_{q_1 q_2 \dots q_{2m+1}, 1} + (m+1) \xi_{11q_2 q_3 \dots q_{2m+1}} \\ &\quad + m \xi_{1q_2 1q_3 \dots q_{2m+1}} \quad \text{and hence} \\ [\xi_1, P_{2m+1}] &= [\xi_1, \xi_{q_1 q_2 \dots q_{2m+1}}]. \end{aligned}$$

Forming in the same way  $\xi_2 P_{2m+1}, \dots, \xi_n P_{2m+1}, P_{2m+1} \xi_2, \dots, P_{2m+1} \xi_n$ , we obtain.

$$(13) \quad P_1 P_{2m+1} = P_{2m+2} - P_{2m+1} + \frac{(n-2m)(2m+3)}{4} P_{2m} = P_{2m+1} P_1.$$

It can be seen similarly that

$$\xi_1 P_{2m} = \xi_{1q_1 q_2 \dots q_{2m}} + \frac{m}{2} \xi_{q_2 \dots q_{2m}} - \frac{m}{2} \xi_{1q_2 \dots q_{2m}} + \frac{m}{2} \xi_{q_2 1q_3 \dots q_{2m}}.$$

and

$$P_{2m} \xi_1 = \xi_{1q_1 q_2 \dots q_{2m}} + \frac{m}{2} \xi_{q_2 \dots q_{2m}} + \frac{m}{2} \xi_{1q_2 \dots q_{2m}} - \frac{m}{2} \xi_{q_2 1q_3 \dots q_{2m}}$$

from which we have

$$[\xi_1, P_{2m}] = m (\xi_{q_2 1q_3 \dots q_{2m}} - \xi_{1q_2 \dots q_{2m}}) = -m [\xi_1, \xi_{q_2 \dots q_{2m}}],$$

i.e.

$$P_{2m} + m P_{2m-1} \text{ is an element of the centre.}$$

[We take  $P_0 = 1$  and  $P_r = 0$  for  $r > n$  and  $r < 0$ ; we notice that there will be  $(k+1)$  elements of this form for  $n = 2k$  or  $2k-1$ .]

We also obtain

$$(14) \quad P_1 P_{2m} = P_{2m+1} + \frac{m(n-2m+1)}{2} P_{2m-1} = P_{2m} P_1.$$

From (13) we have  $P_1^2 = P_2 - P_1 + \frac{3n}{4}$

$$\therefore P_1 + P_2 = P_1^2 + 2P_1 - \frac{3n}{4}.$$

$$\begin{aligned} \therefore (P_1 + P_2) P_{2m} &= P_1 P_{2m+1} + \frac{m(n-2m+1)}{2} P_1 P_{2m-1} \\ &\quad + 2P_1 P_{2m} - \frac{3n}{4} P_{2m}. \end{aligned}$$

$$\begin{aligned} (P_1 + P_2) P_{2m-1} &= P_1 P_{2m} + P_1 P_{2m-1} \\ &\quad + \frac{(n-2m+2)(2m+1)}{4} P_1 P_{2m-2} - \frac{3n}{4} P_{2m-1}. \end{aligned}$$

We thus obtain

$$\begin{aligned} (P_1 + P_2)(P_{2m} + mP_{2m-1}) + \frac{3n}{4}(P_{2m} + mP_{2m-1}) &= \\ P_1 P_{2m+1} + (m+2) P_1 P_{2m} + \frac{m(n-2m+3)}{2} P_1 P_{2m-1} \\ + \frac{m(n-2m+2)(2m+1)}{4} P_1 P_{2m-2} \\ = P_{2m+2} - P_{2m+1} + \frac{(n-2m)(2m+3)}{4} P_{2m} \\ + \frac{m(n-2m+3)}{2} P_{2m} + (m+2) P_{2m+1} \\ + \frac{m(m+2)(n-2m+1)}{2} P_{2m-1} - \frac{m(n-2m+3)}{2} P_{2m-1} \\ + \frac{m(n-2m+3)(n-2m+2)(2m+1)}{8} P_{2m-2} \\ + \frac{m(n-2m+2)(2m+1)}{4} P_{2m-1} \\ + \frac{m(m-1)(2m+1)(n-2m+2)(n-2m+3)}{3} P_{2m-3} \end{aligned}$$

or we obtain the recurrent relation

$$\begin{aligned} (15) \quad (P_1 + P_2)(P_{2m} + mP_{2m-1}) &= (P_{2m+2} + \overline{m+1} P_{2m+1}) + m(n-2m)(P_{2m} + mP_{2m-1}) \\ &\quad + \frac{m(2m+1)(n-2m+3)(n-2m+2)}{8} (P_{2m-2} + \overline{m-1} P_{2m-3}). \end{aligned}$$

It is evident that on utilising (15) in succession, we can express a central element

$$P_{2m} + m P_{2m-1} \text{ as a polynomial in } \theta = P_1 + P_2 = \sum_{r=1}^n \xi_r + \sum_{r,s=1}^n \{ \xi_r, \xi_s \}; r \neq s.$$

We obtain the minimal equation that  $\theta = P_1 + P_2$  satisfies indirectly as follows:

Since  $\text{spur } \xi_r = \text{spur } \xi_s = \text{spur } \{ \xi_r, \xi_s \},$

$\text{spur of } \theta = \frac{n(n+1)}{2} S_{nr}$  in the irreducible representation  $D_{nr},$

i.e. 
$$\text{spur } \theta = \frac{n(n+1)}{2} f_{nr} \frac{n^2 + (5-4r)n + 4(r-1)(r-3)}{2n(n+1)}.$$

Hence the roots of  $\theta$  are 
$$\frac{n^2 + (5-4r)n + 4(r-1)(r-3)}{4}$$

$$r = 1, 2, 3, \dots (k+1) \text{ where } n=2k \text{ or } 2k-1$$

or the minimal equation that  $\theta$  satisfies is

$$(16) \quad \prod_{r=1}^{k+1} \left\{ \theta - \frac{n^2 + (5-4r)n + 4(r-1)(r-3)}{4} \right\} = 0.$$

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# SUMMARY

In this paper we determine explicitly the matrices of all the finite-dimensional representations of the Lie-algebra of the orthogonal group with any number of symbols with spin  $= \frac{3}{2}$ . For this purpose we use the direct product resolution of such an algebra into that of a Dirac algebra and a  $\xi$ -algebra due to Madhava Rao and others. We find first of all the matrices for the representations of the  $\xi$ -algebra; since those of the Dirac-algebra are known one can work out the same for the Lie-algebra. We determine finally the centre of the  $\xi$ -algebra.

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